Virtual invariants of Quot schemes of surfaces.

I. Quot schemes & virtual classes.

II. Multiplicative structural formula & Seiberg–Witten invariants.

III. Rationality of descendent series & virtual Segre/Verlinde series.

<u>Main references</u> : Arbesfeld, Johnson, Lim, Oprea, Pandharipande

- [OP] : Quot schemes of curves and surfaces : virtual classes, integrals, Euler characteristics.
- [JOP] : Rationality of descendent series for Hilbert and Quot schemes of surfaces
- [L] : Virtual χ_{-y}-genera of Quot schemes on surfaces
- [AJLOP]: The virtual K-theory of Quot schemes of surfaces

Goal

- S : smooth projective surface with $p_g(S) > 0$
- $N \geq 1$, $\beta \in H^2(S,\mathbb{Z})_{\geq 0}$ ← fix topological data for moduli
- $\alpha_s \in K^0(S)$, $k_s \geq 0$ for $s = 1, \dots, \ell$ ← descendent insertions & degree

Homological / K-theoretic descendent series of Quot schemes

$$\alpha \left\{ \begin{array}{l} Z^H_{S,N,\beta}(q) := \sum_{n\in\mathbb{Z}} q^n \int_{[\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)]^{vir}} \prod_{s=1}^{\ell} c_{k_s}(\alpha_s^{[n]}) \cdot c(T^{vir}) \\ \\ Z^K_{S,N,\beta}(q) := \sum_{n\in\mathbb{Z}} q^n \chi^{vir}\left(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n), \bigotimes_{s=1}^{\ell} \Lambda^{k_s}\alpha_s^{[n]} \otimes \Lambda_{-y}\Omega^{vir}\right) \end{array} \right.$$

are given by Laurent series expansion of rational functions in the q variable.

I. Quot schemes & virtual classes.

① Grassmannian

$$\begin{cases} V : \text{vector space} /\mathbb{C} \quad (\text{of rank } N) \\ r : \text{integer s.t. } 0 \le r \le N. \end{cases}$$

$\mathrm{Grass}(V, r) := \{ S \mid S \subseteq V \text{ subspace of rank } N-r \}$

Equivalently, it parametrizes quotients

$$\varphi : V \twoheadrightarrow Q \quad \text{s.t.} \quad \mathrm{rank}(Q) = r.$$

Here we identify $[\varphi_1 : V \twoheadrightarrow Q_1] = [\varphi_2 : V \twoheadrightarrow Q_2]$

if $\ker(\varphi_1) = \ker(\varphi_2)$.

We know a lot about $\mathrm{Grass}(V, r) = G$.

(i) Representable by a projective scheme

$$\Rightarrow \exists \text{ universal quotient} : 0 \to S \to \mathcal{O}_G \otimes V \to Q \to 0$$

$$\begin{array}{c} \downarrow \\ G \end{array}$$

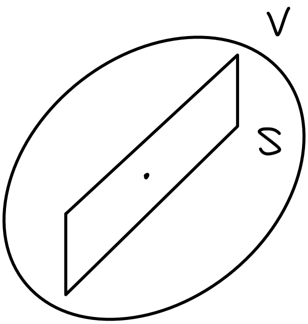
(ii) Deformation theory

At $x = [0 \to S \to V \to Q \to 0] \in G$,

$$T_G\Big|_x = Hom_{\mathbb{C}}(S, Q).$$

(Why?) To infinitesimally deform $S \xhookrightarrow{i} V$,

we need a linear map $f : S \to V$



to make $i + \varepsilon \cdot f : S \hookrightarrow V$.

However, if f factors through S,

then it gives a same subspace.

$\therefore$ We need $\bar{f} : S \to V/S = Q$.

This description glues to give

$$T_G = \mathcal{H}om_{\mathcal{O}_G}(\mathcal{S}, \mathcal{Q}).$$

In particular, G is smooth of

$$\dim = r \cdot (N - r).$$

(iii) Intersection theory.

- Tautological classes : Using $0 \to S \to \mathcal{O}_G \otimes V \to Q \to 0$,

  define : $c_i(S)$, $c_j(Q)$.

- $A^*(G) = H^*(G, \mathbb{Z}) = \dfrac{\mathbb{Z}\left[c_i(S), c_j(Q) : \begin{array}{l} 1 \le i \le N-r \\ 1 \le j \le r \end{array}\right]}{\langle c(S)\, c(Q) = 1 \rangle}$

- $\exists$ additive basis of $H^*(G)$, called Schubert classes

  e.g. $\begin{cases} c_i(S) = (-1)^i \sigma_{1,1,1,\dots,1} \\ c_j(Q) = \sigma_j \end{cases}$

② Grothendieck's Quot schemes

We consider the same construction over a projective scheme.

Let $(X, \mathcal{O}_X(1))$ be a proj. scheme with ample $\mathcal{O}_X(1)$.

Recall we need $\begin{cases} V : \text{vector space} \\ r : \text{integer} \end{cases}$ for $\mathrm{Grass}(V, r)$.

Instead, we fix $\begin{cases} E : \text{coherent sheaf on } X \\ P(m) : \text{Hilbert polynomial.} \end{cases}$

Consider a moduli space $\mathrm{Quot}_{X, \mathcal{O}(1)}(E, P(m))$ parametrizing

$$[\varphi : E \twoheadrightarrow Q] \quad \text{s.t.} \quad \chi(X, Q(m)) = P(m) \quad \forall m \gg 0.$$

Again, $[\varphi_1 : E \twoheadrightarrow Q_1] = [\varphi_2 : E \twoheadrightarrow Q_2]$ if $\ker(\varphi_1) = \ker(\varphi_2)$.

Eg. 1) $X = pt$, $E = V$, $P(m) = r$ $\rightsquigarrow$ $\mathrm{Grass}(V, r)$.

2) $E = \mathcal{O}_X$ $\rightsquigarrow$ $\mathrm{Hilb}_X^{P(m)}$ parametrizes subschemes.

Some properties of Grassmannian also holds for Quot schemes.

(i) Representable by projective scheme.

$\Rightarrow$ $\exists$ universal quotient $\quad 0 \to S \to q^* E \to Q \to 0$

$$\begin{tikzcd}
& \mathrm{Quot} \times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathrm{Quot} & & X
\end{tikzcd}$$

(ii) Deformation theory

At $\quad x = [0 \to S \to E \to Q \to 0] \in \mathrm{Quot}$,

$$T_{\mathrm{Quot}}\big|_x = \mathrm{Hom}_{\mathcal{O}_x}(S, Q).$$

We also have differences!

- $T_{\mathrm{Quot}} \neq \mathcal{H}om_p(S, Q)$.

  Instead, the dual description glues globally

$$\Omega_{\mathrm{Quot}} = \mathcal{E}xt_p^d \, (Q, S \otimes K_X)$$

when X is smooth of dimension d.

- $\dim \mathrm{Hom}(S,Q)$ may depend on $x \in \mathrm{Quot}$.
- $\exists$ obstruction space $\mathrm{Ext}^1_{\mathcal{O}_x}(S,Q)$ and even higher obstructions $\mathrm{Ext}^{\geq 2}_{\mathcal{O}_x}(S,Q)$.

(iii) Intersection theory.

- Tautological classes : $0 \to S \to q^*E \to Q \to 0$

$$p_*\big(c_J(Q) \cdot q^*\alpha\big) \in H^*(\mathrm{Quot})$$

$\Longrightarrow p_*(c_I(S) \cdot c_J(Q) \cdot q^*\alpha)$

$\Longrightarrow$ kunneth components of $c_i(S)$, $c_j(Q)$

- $A(\mathrm{Quot})$, $H^*(\mathrm{Quot})$: Very little is known.

$\therefore$ We instead consider virtual intersection theory when such structure exists.

$$\int_{[\mathrm{Quot}]^{\mathrm{vir}}} \text{tautological classes}$$

③ Virtual class of Quot schemes

* As any other sheaf-theoretic moduli problems, Quot schemes possess a perfect obstruction theory only for low dimensional target X.

* From now, we assume X is smooth projective variety $/\mathbb{C}$.

By Hirzebruch–Riemann–Roch, Chern character

$\mathrm{ch}\,Q = v \in H^{\mathrm{even}}(X)$ determines the Hilbert polynomial

$$\chi(Q(m)) = \int_X v \cdot e^{mH}\,\mathrm{td}(X) \quad \forall m \in \mathbb{Z}.$$

Denote $\mathrm{Quot}_X(E,v) := \{ E \twoheadrightarrow Q \mid \mathrm{ch}\,Q = v \}$.

* $\mathrm{Quot}_X(E,v)$ possesses a perfect obstruction theory if

$$\begin{cases} \mathrm{Tan} = \mathrm{Hom}(S,Q) \\ \mathrm{Ob}_S^1 = \mathrm{Ext}^1(S,Q) \\ \mathrm{Ob}_S^{\geq 2} = \mathrm{Ext}^{\geq 2}(S,Q) = 0 \end{cases} \qquad \forall x = [0 \to S \to E \to Q \to 0] \in \mathrm{Quot}$$

In this case, $T^{\mathrm{vir}}_{\mathrm{Quot}} = R\mathcal{H}om_p(\mathcal{S},\mathcal{Q}) \in K^0(\mathrm{Quot})$.

(i) $\underline{\dim X = 0}$: $\mathrm{Quot} = \mathrm{Grass}$ $\therefore$ smooth without obstruction.

(ii) $\underline{\dim X = 1}$: $\mathrm{Ext}^{\geq 2}(S, Q) = 0$ for dimension reason.

(iii) $\boxed{\dim X = 2}$:

$\boxed{1}$ Suppose that $\begin{cases} E : \text{torsion-free} \\ v = (0, *, *) : \text{torsion class}. \end{cases}$

$$\forall \ [0 \to \underset{\text{torsion-free}}{S} \to E \to \underset{\text{torsion}}{Q} \to 0] \in \mathrm{Quot}_X(E, v),$$

$$\mathrm{Ext}^2(S, Q) = \mathrm{Hom}(Q, S \otimes k_X)^\vee = 0.$$

$\boxed{2}$ Suppose $\begin{cases} \mathrm{Ext}^2(E, E) = 0, \text{ (smooth point on moduli of sheaves)} \\ v : \text{arbitrary} \end{cases}$

$$\forall \ [0 \to S \to E \to Q \to 0] \in \mathrm{Quot}_X(E, v),$$

$$\begin{aligned} \mathrm{Ext}^2(S, Q)^\vee &= \mathrm{Hom}(Q, S \otimes k_X) \\ &\hookrightarrow \mathrm{Hom}(Q, E \otimes k_X) \\ &\hookrightarrow \mathrm{Hom}(E, E \otimes k_X) \\ &= \mathrm{Ext}^2(E, E)^\vee \\ &= 0 \end{aligned}$$

$\ulcorner$ $\mathrm{Quot}_X(\mathcal{E}, v)$

$\downarrow$ relative pot

M_e^{smooth} $\lrcorner$

(iii) $\underline{\dim X = 3}$

Consider $\mathrm{Hilb}_X(\beta, n) = \mathrm{Quot}_X(\mathcal{O}_X, v = (0, 0, \beta, *))$.

Given $[0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0] \in \mathrm{Hilb}_X(\beta, n)$,

$$\left.\begin{aligned} \mathrm{Tan} &= \mathrm{Hom}(I_Z, \mathcal{O}_Z) \\ \mathrm{Obs}^1 &= \mathrm{Ext}^1(I_Z, \mathcal{O}_Z) \\ \mathrm{Obs}^2 &= \mathrm{Ext}^2(I_Z, \mathcal{O}_Z) \end{aligned}\right\} \text{obstruction theory is } \underline{\text{Not}} \text{ perfect.}$$

$$\mathrm{Obs}^3 = \mathrm{Ext}^3(I_Z, \mathcal{O}_Z) = \mathrm{Hom}(\mathcal{O}_Z, I_Z \otimes K_X) = 0$$

However, $\mathrm{Hilb}_X(\beta, n)$ = moduli of ideal sheaves.

This gives another obstruction theory

$$\left.\begin{aligned} \mathrm{Tan} &= \mathrm{Ext}^1(I_Z, I_Z)_0 \\ \mathrm{Obs} &= \mathrm{Ext}^2(I_Z, I_Z)_0 \end{aligned}\right\} \text{perfect obstruction theory.}$$

$$\mathrm{Obs}^{\geq 2} = \mathrm{Ext}^{\geq 3}(I_Z, I_Z)_0 = 0$$

More generally, one consider $\mathrm{Quot}_X(\underset{\text{vector bundle}}{E}, v = (0, 0, \beta, *))$.

Question: Under what condition does $\mathrm{Quot}_X(E, v)$ admit a perfect obstruction theory with

$$T^{vir} = R\mathcal{H}om_p(q^*E, q^*E) - R\mathcal{H}om_p(\mathcal{S}, \mathcal{S}) \ ?$$

$\exists$ partial results for zero dimensional quotients. ($v = (0, 0, 0, n)$)

$\boxed{1}$ $X = \mathbb{C}^3$: [S. Beentjes, A. Ricolfi]

$$\begin{array}{ccc} \mathrm{Quot}_{\mathbb{C}^3}(\mathcal{O}_{\mathbb{C}^3}^{\oplus N}, n) & \hookrightarrow & \mathrm{ncQuot} \\ \| & & \\ \mathrm{Zero}(df) & & \end{array}$$

$\boxed{2}$ X smooth projective 3-fold : [A. Ricolfi]

<u>case 1</u> : $\begin{cases} \circ \ H^{>0}(X, \mathcal{O}_X) = 0 \\ \circ \ \mathrm{Hom}(E, E) = \mathbb{C}, \ \mathrm{Ext}^{>0}(E, E) = 0 \end{cases}$

<u>case 2</u> : $\begin{cases} \circ \ X : \text{CY-3fold} \quad (H^1(\mathcal{O}_X) = 0) \\ \circ \ \mathrm{Hom}(E, E) = \mathbb{C}, \ \mathrm{Ext}^1(E, E) = 0 \end{cases}$

(iv) <u>$\dim X = 4$</u>

$\boxed{1}$ X CY 4-fold : $\mathrm{Hilb}_X(\beta, n)$ possesses Oh–Thomas class.

$\boxed{2}$ $X = \mathbb{C}^4$: [M. Kool, J. Rennemo]

$$\begin{array}{ccc} & & (V, q) \\ & & \downarrow \uparrow s : \text{isotropic.} \\ \mathrm{Quot}_{\mathbb{C}^4}(\mathcal{O}_{\mathbb{C}^4}^{\oplus N}, n) & \hookrightarrow & \mathrm{ncQuot} \\ \| & & \\ \mathrm{Zero}(s) & & \end{array}$$

④ Applications of study of $\mathrm{Quot}_C(\mathcal{O}_C^{\oplus N}, \underbrace{r, d}_{})$.

Q : rank r, deg d.

This has been extensively studied by series of works by A. Marian, D. Oprea.

(i) Applications to $M_C(r,d)$ with $(r,d)=1$.

$\hookrightarrow$ stable rank r, deg d vector bundles.

We may assume $d \gg 0$, because $M_C(r,d) \xrightarrow[\sim]{\otimes \mathcal{O}_C(pt)} M_C(r, d+r)$.

$$\begin{array}{ccc} \mathcal{E} & & \\ \downarrow & & \\ C \times M_C(r,d) & & p_*\mathcal{E} \\ & \searrow^{p} & \downarrow \\ & & M_C(r,d) \end{array}$$

$p_*\mathcal{E}$: vector bundle b/c $H^1(C, E) = 0$ $\forall [E] \in M_C(r,d)$.

Note that $\mathbb{P}((p_*\mathcal{E})^{\oplus N})$ parametrizes $\left[\mathcal{O}_C^{\oplus N} \xrightarrow{[\phi]} E\right]$

where
- E stable
- $[\phi]$ non zero, up to $\mathbb{C}^\times$.

<u>Remark</u> : $\mathbb{C}^\times = \mathrm{Aut}(E)$.

$$\Longrightarrow \quad \mathrm{Quot}_C(\mathcal{O}_C^{\oplus N}, r, d) \overset{\text{bir}}{\dashleftarrow\dashrightarrow} \mathbb{P}((p_*\mathcal{E})^{\oplus N})$$

$$\begin{array}{ccc} \cup\text{ open} & & \cup\text{ open} \\ \{\mathcal{O}_C^{\oplus N} \twoheadrightarrow Q \mid Q : \text{stable}\} & = & \{\mathcal{O}_C^{\oplus N} \xrightarrow{[\phi]} E \mid \phi : \text{surjective}\} \end{array}$$

If $d \gg 0$, then $\mathrm{Quot}_C(\mathcal{O}_C^{\oplus N}, r, d)$ is $\begin{cases} \text{irreducible,} \\ \text{generically smooth,} \\ \text{of expected dim.} \end{cases}$

Up shot

$\boxed{1}$ Certain intersection numbers of $M_C(r,d)$ can be calculated via virtual intersection numbers of $\mathrm{Quot}_C(\mathcal{O}_C^{\oplus N}, r, d)$ using the diagram

$$\begin{array}{ccc} \mathrm{Quot}_C(\mathcal{O}_C^{\oplus N}, r, d) & \dashleftarrow\dashrightarrow & \mathbb{P}((p_*\mathcal{E})^{\oplus N}) \\ & & \downarrow \\ & & M_C(r,d) \end{array}$$

$\boxed{2}$ Virtual intersection theory of $\mathrm{Quot}_C(\mathcal{O}_C^{\oplus N}, r, d)$ can be effectively studied using virtual localization w.r.t.

$$(\mathbb{C}^\times)^N \curvearrowright \mathrm{Quot}_C(\mathcal{O}_C^{\oplus N}, r, d).$$

e.g. fixed loci are of the form $C^{[d_1]} \times \cdots \times C^{[d_{N-r}]}$.

(ii) $\mathrm{Quot}_C(\mathcal{O}_C^{\oplus N}, r, d)$ provides compactification of $\mathrm{Map}_C(G(N,r), d)$.

This follows from universal property of $G(N,r)$:

$$\left\{ \begin{array}{c} C \longrightarrow G(N,r) \\ \text{of degree } d \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathcal{O}_C^{\oplus N} \twoheadrightarrow Q \quad \text{s.t.} \\ \mathrm{rk}\, Q = r,\ \deg Q = d \\ \text{\& } Q : \text{vector bundle} \end{array} \right\}$$

$\therefore$ $\mathrm{Quot}_C(\mathcal{O}_C^{\oplus N}, r, d)$ provides compactification by allowing Q to degenerate to coherent sheaves.

Indeed, one can study "Gromov invariants" via Quot schemes. (Bertram, Daskalopoulos, Wentworth).

This idea also grows to the study of stable quotients by Marian, Oprea, Pandharipande.

(iii) Proof of strange duality by Marian, Oprea.

This was the highlight of the study of Quot schemes of curves using ideas of (i), (ii).

⑤ Generating series of virtual invariants of Quot schemes of surfaces

- S : smooth projective surface
- $N \geq 1$, $\beta \in H_2(S,\mathbb{Z})_{\geq 0}$, $n \in \mathbb{Z}$

$$\mathrm{Quot}_S(\mathbb{C}^N,\beta,n) := \left\{ [0 \to S \to \mathcal{O}_S^{\oplus N} \to Q \to 0] \;\middle|\; \begin{array}{l} \mathrm{rk}\, Q = 0 \\ c_1(Q) = \beta \\ \chi(Q) = n \end{array} \right\}$$

1) Tautological class :

$$\alpha \in K^0(S) \rightsquigarrow \alpha^{[n]}_{N,\beta} := p_!\left(Q \cdot q^* \alpha\right) \in K^0(\mathrm{Quot}).$$

2) Multiplicative genus :

Given $f \in \left(F[\![z]\!]\right)^\times$ for some field F, define

$$f : K^0(M) \longrightarrow H^*(M,F)$$

$$V \longmapsto \prod_{i \in I} f(z_i) \quad \text{where } z_i : \text{Chern roots}.$$

e.g.

$$f(z) = 1 + x \cdot z \rightsquigarrow f(\alpha) = c_x(\alpha).$$

$$f(z) = \frac{z}{1-e^{-z}} \rightsquigarrow f(\alpha) = td(\alpha).$$

Given $\alpha_1, \cdots, \alpha_\ell \in K^0(S)$, $f_1, \cdots, f_\ell, g \in (F[\![z]\!])^\times$, define

$$Z^{\underline{f}, g}_{S,N,\beta}(q \mid \underline{\alpha}) := \sum_{n \in \mathbb{Z}} q^n \int_{[\mathrm{Quot}_S(\mathbb{C}^N, \beta, n)]^{\mathrm{vir}}} \prod_{s=1}^{\ell} f_s(\alpha_s^{[n]}) \cdot g(T^{\mathrm{vir}}_{\mathrm{Quot}})$$

In the next lecture, we study the structure of these generating series.

Thank You